NORTH-HOLLAND

Algorithms for Weakly Nonnegative Quadratic Forms

A. Dean
Department of Mathematics
Bishop's University
Lennoxville, Quebec, Canada

and

J. A. de la Peña
Instituto de Matemáticas, UNAM
Ciudad Universitaria
México 04510, D.F., Mexico

Submitted by Richard A. Brualdi

ABSTRACT

Let $q: \mathbf{Z}^n \to \mathbf{Z}$ with $q(v) = \sum_{i=1}^{n} v(t)^2 + \sum_{i<j} a_{ij} v(i)v(j)$ be a unit form. We present an algorithm that allows one to check if q is weakly nonnegative [i.e., $q(v) \geqslant 0$ for any vector $v \in \mathbf{N}^n$]. The algorithm also calculates the set of critical vectors of q. We sketch the relation of this problem to the representation theory of finite-dimensional algebras.

INTRODUCTION

In this work we consider integral quadratic forms $q: \mathbf{Z}^n \to \mathbf{Z}$ with $q(v) = \sum_{i=1}^{n} v(i)^2 + \sum_{i<j} a_{ij} v(i)v(j)$. Such a quadratic form is called a *unit form*. The associated symmetric matrix $A_q = (a_{ij})$ has $a_{ii} = 2$.

A unit form q is said to be *weakly positive* [*weakly nonnegative*] if for every vector $0 \neq v \in \mathbf{Z}^n$ with nonnegative coordinates we have $q(v) > 0$ [$q(v) \geqslant 0$]. A first obvious problem is the following: given a unit form q, how does one check whether or not q is weakly positive or weakly nonnegative? In [8], the known answers for the weakly positive case are surveyed. In

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particular, a quite efficient algorithm is presented; this algorithm is already implemented for a computer [1]. Some algorithms are known for the weakly nonnegative case [8, 12]. In [2] other implementations are reported.

The first purpose of this work is to present an algorithm to decide whether or not a given unit form q is weakly nonnegative. Moreover, in case q is weakly nonnegative, this algorithm provides a set $\{v_1, \dots, v_s\}$ of vectors with nonnegative coordinates in $\mathbf{Z}^n$ satisfying: (1) $q(v_i) = 0$, $i = 1, \dots, s$; (2) for any $v \in \mathbf{Z}^n$ with nonnegative coordinates and $q(v) = 0$, there is a linear combination $v = \sum_{i=1}^{s} \mu_i v_i$ with $\mu_i \in \mathbf{Q}^+$ (= nonnegative rational numbers). These vectors allow us to calculate several invariants of q.

At the end, we give a brief survey of our main motivation: the representation theory of finite-dimensional algebras over a field k. For such an algebra $\Lambda = k[Q]/I$ (for notation, see [5, 8, 11]) such that the quiver Q has no oriented cycles, a unit form (the so called *Tits form*) q_Λ is defined. If Λ is of tame representation type, then q_Λ is weakly nonnegative; the converse holds in several situations. In many cases, the nonnegative vectors $v \neq 0$ with $q_\Lambda(v) = 0$ are associated with one-parameter families of indecomposable modules. These motivations will allow us to present some examples related to Sections 1 and 2.

1. BASIC FACTS

1.1

Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a *unit form*, that is, for each $v \in \mathbf{Z}^n$

$$q(v) = \sum_{i=1}^{n} v(i)^2 + \sum_{i<j} a_{ij} v(i) v(j)$$

where a_{ij} are integral numbers. The symmetric matrix $A_q = (a_{ij})$ with $a_{ii} = 2$ defines a bilinear form $q(x, y) = x A_q y^t$. In particular, $q(v) = \frac{1}{2} q(v, v)$.

For a vector $v \in \mathbf{Q}^n$ with nonnegative coordinates, we write $v \geqslant 0$. If all the coordinates are strictly positive, we write $v \gg 0$.

Recall that we say that the unit form q is weakly positive [weakly nonnegative] if for every $0 \neq v \geqslant 0$ we have $q(v) > 0$ [$q(v) \geqslant 0$].

We define the set Σ_q^1 of *positive roots* of q,

$$\Sigma_q^1 = \{v \in \mathbf{Z}^n : 0 \leqslant v \text{ and } q(v) = 1\}$$

and the set of *null roots* Σ_q^0,

$$\Sigma_q^0 = \{v \in \mathbf{Z}^n : 0 \leqslant v \text{ and } q(v) = 0\}.$$

1.2

By a result of Drozd [4] (see also [12]), a weakly positive unit form q has a finite set Σ_q^1 of positive roots.

For a set $J = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ of indices, we denote by $s_J : \mathbf{Z}^m \to \mathbf{Z}^n$ the inclusion such that $s_J(e_{i_j}) = e_j$, where $\{e_i, \ldots, e_n\}$ denotes the canonical basis of $\mathbf{Z}^n$. Then $q^J = qs_J$ is the restriction of q to J. If $J = \{1, \ldots, n\} \setminus \{j\}$, we denote $q^{(j)} := q^J$.

A unit form q is called *critical* if all the restrictions $q^{(j)}$, $1 \leqslant j \leqslant n$ are weakly positive but q itself is not. By a result of Ovsienko [7], if q is critical and $n \geqslant 3$, then q is nonnegative and there is a unique vector $0 \ll z \in \mathbf{Z}^n$ such that $\Sigma_q^0 = \mathbf{N}^+ z$. A vector z with this property is called a *critical vector*. In case $n = 2$, we call the vector $(1, 1)$ critical; observe that $q(1, 1) \leqslant 0$ [but possibly $q(1, 1) < 0$].

Let q be a unit form and q^J be a critical restriction. Let $0 \ll z \in \mathbf{Z}^m$ be a critical vector of q^J; then the vector $s_J(z) \in \Sigma_q^0$ is called a critical vector of q. A critical vector v has coordinates $0 \leqslant v(i) \leqslant 6$ for $i = 1, \ldots, n$.

Clearly, a unit form q is weakly positive if and only if it has no critical restrictions if and only if $q(v) > 0$ for any $0 \neq v \in ([0, 6] \cap \mathbf{Z})^n$.

Finally, we recall that the critical forms were classified by von Höhne [6].

1.3

Consider the bilinear form $q(-,-)$ associated to the unit form $q : \mathbf{Z}^n \to \mathbf{Z}$.

LEMMA. *Let* $1 \leqslant i \leqslant n$ *and* $0 \leqslant v \in \mathbf{Z}^n$.

(a) *Assume that* q *is weakly positive and* $v \in \Sigma_q^1$. *Then* $-1 \leqslant q(v, e_i)$. *If moreover* $i \in \operatorname{supp}(v)$ [$:= \{1 \leqslant j \leqslant n : v(j) \neq 0\}$], *then* $q(v, e_i) \leqslant 1$.

(b) *Assume that* q *is weakly nonnegative and* $v \in \Sigma_q^1$. *Then* $-2 \leqslant q(v, e_i)$. *If moreover* $i \in \operatorname{supp}(v)$, *then* $q(v, e_i) \leqslant 2$.

(c) *Assume that* q *is weakly nonnegative and* $v \in \Sigma_q^0$. *Then* $0 \leqslant q(v, e_i)$. *If moreover* $i \in \operatorname{supp}(v)$, *then* $q(v, e_i) \leqslant 0$.

Proof. This is very elementary. Anyway, (a) is shown in [12]; (b) is similar. For (c), if $q(v, e_i) < 0$, then $q(2v + e_i) = 1 + 2q(v, e_i) < 0$, which contradicts the weak nonnegativeness of q. Similarly if $i \in \operatorname{supp}(v)$, then $0 \leqslant q(2v - e_i)$ implies that $q(v, e_i) \leqslant 0$. ∎

1.4

The following criterion for weakly nonnegativeness was essentially proved in [9].

PROPOSITION. *Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a unit form. The following conditions are equivalent*:

(i) *q is weakly nonnegative.*
(ii) *For every $v \in \Sigma_q^0$ one has $vA_q \geqslant 0$.*
(iii) *For every critical vector v of q, one has $vA_q \geqslant 0$.*

Proof. That (i) implies (ii) is part (c) of the last lemma. That (ii) implies (iii) is trivial. Assume that (iii) is true and q is not weakly nonnegative. We may suppose that q is chosen with minimal possible n. Let $0 \leqslant z$ be a vector with $q(z) < 0$. Then $0 \ll z$. Since q is not weakly positive, we find a critical restriction q^J. Let $0 \leqslant v$ be the corresponding critical vector of q. We may find a rational number $a < 0$ and some $1 \leqslant j \leqslant n$ such that $0 \leqslant z + av$ and $(z + av)(j) = 0$. Then (iii) is satisfied for $q^{(j)}$, but $q^{(j)}(z + av) < avA_q z^t \leqslant 0$, contradicting the minimality of n. ∎

1.5

Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a unit form. The linear transformation $\sigma_i: \mathbf{Z}^n \to \mathbf{Z}$ defined by $\sigma_i(v) = v - q(v, e_i)e_i$ is called *reflection* with respect to e_i. The transformation σ_i satisfies $\sigma_i^2 = \mathrm{id}$ and $q(\sigma_i(v)) = q(v)$ for every $v \in \mathbf{Z}^n$.

The following result gives an inductive construction of the null roots of a critical form. This will be useful later.

LEMMA. *Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a critical unit form with $n \geqslant 3$. Let $0 \neq u \in \Sigma_q^0$. Then there is a vector $v \in \Sigma_q^1$ and an index j such that $q(v, e_j) = -2$ and $u = v + e_j$. Moreover, if u is critical, there is a sequence of indices $i_1, \dots, i_m$, t and roots*

$$v = v_0 > v_1 > \cdots > v_m = e_t$$

such that $v_p = \sigma_{i_p}(v_{p-1}) = v_{p-1} - e_{i_p}$, $p = 1, \dots, m$.

Proof. Let j be any index such that $v = u - e_j \geqslant 0$. Then

$$q(v) = q(u, e_j) + 1 = 1 \quad \text{and} \quad q(v, e_j) = -q(e_j, e_j) = -2.$$

Assume that u is critical. Since

$$2 = q(v,v) = \sum_{i=1}^{n} v(i) q(v,e_i),$$

there is some i with $v(i) \neq 0 < q(v, e_i)$. By Section 1.3, $q(v, e_i) \leqslant 2$. If $q(v, e_i) = 2$, we may consider $0 \leqslant w = v - e_i$ with

$$q(w) = 2 - q(v,e_i) = 0.$$

Since q is critical, then $w = mu$ for some natural number m. But this is impossible, since $0 \leqslant u - w \neq 0$. Hence $q(v, e_i) = 1$ and $v_1 = \sigma_i(v) = v - e_i \in \Sigma_q^1$. We can continue by induction. ■

2. THE ALGORITHM

2.1

Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a unit form. Starting with the set $C_1 = \{e_i: 1 \leqslant i \leqslant n\}$, we will define inductively a procedure for constructing a new set $C_{s+1} \subset \Sigma_q^1$ from $C_s \subset \Sigma_q^1$. The procedure may fail; in that case, C_{s+1} is not defined and the procedure stops, indicating that q is not weakly nonnegative. Otherwise, it continues. More precisely:

(1) Define $C_1 = \{e_i: 1 \leqslant i \leqslant n\}$.

(2) Assume that $C_s = \{v_1, \dots, v_m\} \subset \Sigma_q^1$ is well defined and the procedure has not failed. Let $v_j \in C_s$; if either

(a) there is some $1 \leqslant i \leqslant n$ such that $q(v_j, e_i) \leqslant -3$, or

(b) there is some $1 \leqslant i \leqslant n$ such that $q(v_j, e_i) = -2$ and not $(v_j + e_i)A_q \geqslant 0$,

then the procedure is said to *fail*. Assume the procedure does not fail.

(3) Let $R_s = \{v \in C_s: v(i) \leqslant 6$ for $1 \leqslant i \leqslant n$ and $q(v, e_j) = -1$ for some $j\}$.

(4) If $R_s = \varnothing$, then $C_{s+1} := \varnothing$ and the process is said to be *successful*.

(5) If $R_s \neq \varnothing$, then

$$C_{s+1} := \left\{\sigma_j(v): v \in R_s \text{ and } q(v,e_j) = -1\right\}.$$

We may immediately observe the following features:

(i) For each $v \in C_s$, the weight $|v| = \sum_{i=1}^{n} v(i) = s \leqslant 6n + 1$.
(ii) Either the procedure fails, or it is successful after at most $6n + 1$ steps.

2.2

THEOREM. *Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a unit form. Then q is weakly nonnegative if and only if the procedure in Section* 2.1 *applied to q is successful.*

Proof. Suppose that q is weakly nonnegative. Assume that the procedure is not successful. Suppose it fails after forming the set $C_s \subset \Sigma_q^1$. Then we find some $v \in C_s$ and either (a) there is some $1 \leqslant i \leqslant n$ with $q(v, e_i) \leqslant -3$ or (b) there is some $1 \leqslant i \leqslant n$ with $q(v, e_i) = -2$ and not $(v + e_i)A_q \geqslant 0$. In case (a), the lemma in Section 1.3 implies that q is not weakly nonnegative. In case (b), we have $q(v + e_i) = 2 + q(v, e_i) = 0$ and the proposition in Section 1.4 implies that q is not weakly nonnegative. This contradiction shows that the procedure is successful.

Assume the process is successful. We have to show that for every critical vector $u \in \Sigma_q^0$, we have $uA_q \geqslant 0$ (Section 1.4). So let $u \in \Sigma_q^0$ be a critical vector. Consider m, the cardinality of supp(u). If $m = 2$ and $q(u) < 0$, then $u = e_i + e_j$ with $q(e_i, e_j) \leqslant -3$. Therefore the procedure should have failed already in step 2. Hence, if $m = 2$, then $u = e_i + e_j$ with $q(e_i, e_j) = -2$. If $m \geqslant 3$, then by Section 1.5 we may write $u = v + e_j$ with $v \in \Sigma_q^1$ and $q(v, e_j) = -2$. Moreover, there is a sequence $i_1, \dots, i_m$ of indices such that $e_l < \sigma_{i_1} e_l < \sigma_{i_2}\sigma_{i_1} e_l < \cdots < \sigma_{i_m} \cdots \sigma_{i_1} e_l = v$ with $\sigma_{i_s}\sigma_{i_{s-1}} \cdots \sigma_{i_1} e_l \in C_{s+1}$ for $0 \leqslant s \leqslant m$. Then $v \in C_{m+1}$. Since $q(v, e_j) = -2$, then $uA_q = (v + e_j)A_q \geqslant 0$ and we are done. ■

REMARKS.

(1) If the procedure for $q: \mathbf{Z}^n \to \mathbf{Z}$ fails, then it fails in at most $\max\{2n - 4, 30\}$ steps. For, assume that q is not weakly nonnegative, and let v be a critical vector such that $vA_q e_j < 0$ for some $1 \leqslant j \leqslant n$. Let m be the cardinality of $J = \text{supp}(v)$. If $m = 2$, then $v = e_i + e_j$ and $|v| = 2$. Therefore the procedure fails in the second step. Assume $m \geqslant 3$. Then [6] shows that there is a linear transformation $T: \mathbf{Z}^n \to \mathbf{Z}^n$ such that $q^J = q_\Delta T$ with q_Δ the critical form associated to a diagram of Euclidean (= extended Dynkin) type; moreover, if u is the critical vector corresponding to q_Δ, then $v(i) \leqslant u(i)$ for any $i \in J$. If Δ is of type $\tilde{\mathbf{A}}_m$, then $|v| \leqslant |u| = m \leqslant n$; if Δ is of type $\tilde{\mathbf{D}}_m$,

then $|v| \leqslant |u| = 2m - 4 \leqslant 2n - 4$; if Δ is of type $\tilde{\mathbf{E}}_p$ ($p = 6, 7$, or 8), then $|v| \leqslant |u| \leqslant 30$.

(2) If the procedure fails, then one is able to obtain explicitly a vector $v \geqslant 0$ such that $q(v) < 0$. Indeed, suppose the procedure fails after s steps. Then for some $v_j \in C_s$ we have either (a) or (b) in the proof holding. If (a) holds, then take $v = v_j + e_i$. If (b) holds, then $(v_j + e_i)A_q e_l^t < 0$ for some $1 \leqslant l \leqslant n$. In this case take $v = 2(v_j + e_i) + e_l$.

(3) The above algorithm has been implemented for a computer [1].

2.3

Of particular interest is that as a by-product of the above procedure, if q is weakly nonnegative, we have constructed all the critical vectors of q. More precisely, let $q'\colon \mathbf{Z}^n \to \mathbf{Z}$ be a weakly nonnegative form, and let $C_1, \ldots, C_s, C_{s+1} = \varnothing$ be the sets defined by the procedure above. For each $1 \leqslant j \leqslant s$, define $N_i = \{v + e_j : v \in C_i \text{ and } q(v, e_j) = -2\}$. Then we have the following:

LEMMA. *Let u be a critical vector (in Σ_q^0). Then $u \in \bigcup_{i=1}^s N_i$.*

Proof. Let m be the cardinality of supp(u). If $m = 2$, then $u = e_i + e_j$ with $q(e_i, e_j) = -2$ [since q is weakly nonnegative, $q(u) = 0$]. Hence $u \in N_2$.

If $m \geqslant 3$, this is a straightforward application of Section 1.5. ∎

2.4

For a weakly nonnegative unit form $q\colon \mathbf{Z}^n \to \mathbf{Z}$, the critical vectors are the smallest pieces necessary to construct null vectors in Σ_q^0, as the following result shows.

PROPOSITION. *Let $q\colon \mathbf{Z}^n \to \mathbf{Z}$ be a weakly nonnegative unit form. Let $\{v_1, \ldots, v_s\}$ be the set of all critical vectors of q. For any $v \in \Sigma_q^0$, there exist numbers $\mu_1, \ldots, \mu_s \in \mathbf{Q}^+$ such that $v = \sum_{i=1}^s \mu_i v_i$.*

Proof. Let $v \in \Sigma_q^0$, and let m be the cardinality of supp(v). We proceed by induction on m. Since $q(v) = 0$, there exists some critical restriction q^J with $J \subset$ supp(v). Let v_J be the critical vector associated to q^J. If m is minimal, then $J =$ supp(v) and $v = av_J$ for some $a \in \mathbf{N}$. We are done.

Otherwise, there is some linear combination $0 \leqslant w = av - bv_j$ with $a, b \in \mathbf{N}$ such that $\mathrm{supp}(w) \subsetneq \mathrm{supp}(v)$. We apply the induction hypothesis to w and obtain the desired expression for v. ■

2.5

There are other important invariants associated with a unit form q which we may calculate with the help of our algorithm. We introduce some definitions.

Let $q: \mathbf{Z}^n \to \mathbf{Z}$ be a unit form. We extend Σ_q^0 to the rational numbers: $\Sigma_q^0 \subset \mathrm{rad}_q^+ := \{v \in \mathbf{Q}^n : 0 \leqslant v \text{ and } q(v) = 0\}$. A subset $V \subset \mathrm{rad}_q^+$ is said to be a *half space* if for every $v, w \in V$ and $\alpha, \beta \in \mathbf{Q}^+$ one has $\alpha v + \beta w \in V$. The dimension of V is the maximal number of linearly independent vectors in V.

A half space $V \subset \mathrm{rad}_q^+$ is said to be *connected* if $\mathrm{supp}\, V = \bigcup_{v \in V} \mathrm{supp}(v)$ is connected in the following sense: for each pair $i, j \in \mathrm{supp}\, V$ there is a sequence $i = i_0, i_1, \ldots, i_m = j$ of indices in $\mathrm{supp}\, V$ with $a_{i_s i_{s+1}} \neq 0$ for $s = 0, \ldots, m-1$. The corank of q, corank q, is by definition the maximal dimension of a connected half space in rad_q^+.

PROPOSITION. *Let q: $\mathbf{Z}^n \to \mathbf{Z}$ be a weakly nonnegative unit form. Let $\{v_1, \ldots, v_m\}$ be the set of all critical vectors of q. Then*:

(a) *Let $V \subset \mathrm{rad}_q^+$ be a half space. Then there is a subset $\{i_1, \ldots, i_t\} \subset \{1, \ldots, s\}$ such that $V \subset \Sigma_{p=1}^t \mathbf{Q}^+ v_{i_p} \subset \mathrm{rad}_q^+$.*

(b) *Let $\{i_1, \ldots, i_t\} \subset \{1, \ldots, s\}$ be a set with maximal cardinality t such that*

(i) *For each $j, l \in \{i_1, \ldots, i_t\}$, $q(v_j, v_l) = 0$;*
(ii) *$\bigcup_{p=1}^t \mathrm{supp}(v_{i_p})$ is connected;*
(iii) *$\{v_{i_1}, \ldots, v_{i_t}\}$ is linearly independent*

Then corank $q = t$.

Proof. (a): Let $L \subset \{1, \ldots, s\}$ be the set of indices defined as follows: $j \in L$ if there is an element $0 \neq v \in V$ and a linear combination $v = \Sigma_{i=1}^s \mu_i v_i$ with $\mu_i \in \mathbf{Q}^+$ and $\mu_j \neq 0$. Clearly, $V \subset \Sigma_{j \in L} \mathbf{Q}^+ v$ To show that $\Sigma_{j \in L} \mathbf{Q}^+ v_j \subset \mathrm{rad}_q^+$, it is enough to show that given $v = \Sigma_{i=1}^s \mu_i v_i$

and $w = \sum_{i=1}^{s} \lambda_i v_i$ in V with $\mu_i, \lambda_i \geqslant 0$ $(1 \leq i \leq n)$ and $\mu_j, \lambda_l > 0$, then $v_j + v_l \in \mathrm{rad}_q^+$, that is $q(v_j, v_l) = 0$. To show this, calculate

$$0 = q(v, w) = \sum_{i,t=1}^{s} \mu_i \lambda_t q(v_i, v_t) \geq \mu_j \lambda_l q(v_j, v_l),$$

because $q(v_i, v_t) \geqslant 0$ for each $1 \leqslant i, t \leq s$. Hence $q(v_j, v_l) = 0$.

(b): Let V be a connected half space of maximal dimension. By (a), we find $L \subset \{1, \dots, s\}$ such that $V \subset W = \sum_{j \in L} \mathbf{Q}^+ v_j \subset \mathrm{rad}_q^+$, and by construction $\operatorname{supp} V = \operatorname{supp} W$. Hence W is a connected half space and $V = W$. Therefore, $\operatorname{corank} q = \dim W \leqslant t$ as given in the definition. On the other hand, the set $\{i_1, \dots, i_t\} \subset \{1, \dots, n\}$ defines a connected half space $\sum_{p=1}^{t} \mathbf{Q}^+ v_{i_p} =: W$ [using (i) and (ii)]. Property (iii) implies that $t = \dim W \leqslant \operatorname{corank} q$. ■

3. EXAMPLES AND MOTIVATION

3.1

For the fundamental definitions needed to understand this section, the reader is referred to [4, 9, 12].

Let $\Lambda = k[Q]/I$ be a basic, finite-dimensional algebra over an algebraically closed field k. We assume that the quiver Q of Λ is connected and without oriented cycles. Assume that the set of vertices Q_0 of Q is $\{1, \dots, n\}$. Associated with Λ we find a unit form $q_\Lambda : \mathbf{Z}^n \to \mathbf{Z}$ called the Tits form of Λ and defined by

$$q_\Lambda(v) = \sum_{i=1}^{n} v(i)^2 - \sum_{(i \to j) \in Q_1} v(i)v(j) + \sum_{i,j} r(i,j)v(i)v(j),$$

where Q_1 is the set of arrows of Q and the number $r(i, j)$ is defined as the cardinality of $L \cap I(i, j)$, where L is a minimal set of generators of I with $L \subset \bigcup_{i,j} I(i, j)$.

The Tits form has played an important role in the representation theory of finite-dimensional algebras (see for example [9]).

3.2

An algebra Λ is said to be of *tame-representation type* if for every $d \in \mathbf{N}$ there are a finite number of $\Lambda - k[t]$-bimodules $M_1, \dots, M_s$ which are free over $k[t]$ and such that up to isomorphism almost every indecomposable Λ-module of dimension d is of the form $M_i \otimes_{k[t]} k[t]/(t - \lambda)$ for some $1 \leqslant i \leqslant s$ and some $\lambda \in k$. We write $\mu_\Lambda(d) = s$ for the minimal possible s in this definition.

THEOREM [8]. *Let Λ be a tame algebra as above. Then the Tits form q_Λ is weakly nonnegative.* ∎

For several important classes of algebras the converse of this result holds. Our motivation to decide whether or not a given unit form is weakly nonnegative comes from these results.

3.3

A tame algebra Λ is said to be of *polynomial growth* if $\mu_\Lambda(d) \leqslant d^m$ for some $m \in \mathbf{N}$ and all d. If $\mu_\Lambda(d) \leqslant c$ for a fixed c and all $d \in N$, then Λ is said to be *domestic*. Domestic algebras are rather well understood, and there is investigation going on about polynomial-growth algebras.

Since we do not want to introduce technical definitions concerning separating properties, we state the next result for a quite particular class of algebras (it holds for the so called completely separating algebras—see [3] and [11]). An algebra $\Lambda = k[Q]/I$ is said to be a *tree algebra* if the quiver Q is a tree (i.e., the underlying graph of Q has no cycles).

THEOREM [11]. *Let $\Lambda = k[Q]/I$ be a tree algebra. Then*:

(a) *If Λ is tame but not of polynomial growth, then* corank $q_\Lambda \geqslant 2$.

(b) *If Λ is a tame polynomial-growth algebra and there exists a sincere indecomposable module (X sincere: $\{i \in Q_0\colon X(i) \neq 0\} = Q_0$), then* corank $q_\Lambda \leqslant 2$.

(c) *The algebra Λ is tame domestic if and only if q_Λ is weakly nonnegative and* corank $q_\Lambda \leqslant 1$.

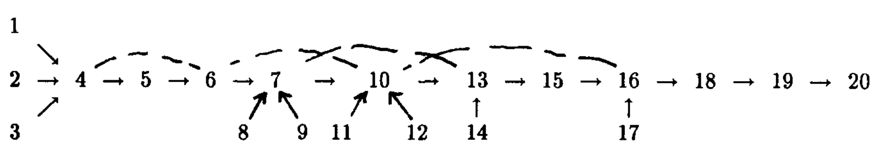
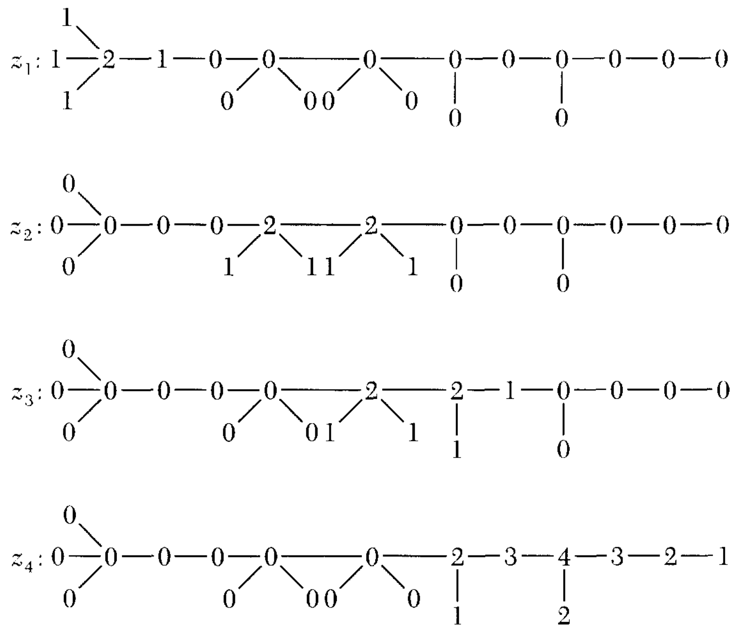


FIG. 1

3.4

We give an example which illustrates all that has been discussed. Consider the algebra $\Lambda = k[Q]/I$ given by the quiver with relations given in Figure 1, where the dashed lines indicate the generators of the ideal I. By techniques of representation theory, we know that Λ is tame but not of polynomial growth. Hence the Tits form q_Λ is weakly nonnegative and corank $q_\Lambda \geqslant 2$. Let us see this with more precision. Applying the algorithm, we check that q_Λ is weakly nonnegative and find the following critical vectors:



where in the place of each vertex we have indicated the value of the vector. We see that $q_\Lambda(z_i, z_j) = 0$ for $1 \leqslant i, j \leqslant 3$, $q_\Lambda(z_1, z_4) = 0$, $q_\Lambda(z_2, z_4) = 8$, $q_\Lambda(z_3, z_4) = 4$. Moreover the half spaces $\mathbf{Q}^+z_1 + \mathbf{Q}^+z_2 + \mathbf{Q}^+z_3$ and $\mathbf{Q}^+z_1 + \mathbf{Q}^+z_4$ are not connected.

Since $\mathbf{Q}^+z_2 + \mathbf{Q}^+z_3$ is connected, then corank $q_\Lambda = 2$.

As another application, we see that for any quotient algebra $\Lambda_i = \Lambda/(i)$ with $10 \leqslant i \leqslant 14$, the Tits form is weakly nonnegative and corank $q_{\Lambda_i} = 1$. Therefore all the algebras Λ_i are domestic.

REMARK. A half space $V \subset \mathrm{rad}^+_{q_\Lambda}$ is connected if and only if $\mathrm{supp}\, V$ yields a full connected subquiver of Q.

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REFERENCES

1 A. Dean, K. Denver, and K. Pershall, Integral Quadratic Forms, an Algorithmic Approach (PASCAL programs: Version 1, July 1992; Version 2, March 1993), Bishop's Univ., Lennoxville, Quebec, Canada.
2 P. Dräxler, Computational aspects in the representation theory of finite dimensional algebras, Preprint, Univ. Bielfeld, 1992.
3 P. Dräxler, Completely separating algebras, *J. Algebra*, 165 (1994), 550–565.
4 Ju. A. Drozd, Coxeter transformations and representations of partially ordered sets, *Funct. Anal. Appl.* 8:219–225 (1974).
5 P. Gabriel, Auslander-Reiten sequences and representation-finite algebras, in *Lecture Notes in Math.* 831, Springer-Verlag, 1980, pp. 1–71.
6 H. J. von Höhne, On weakly positive unit forms, *Comment. Math. Helv.* 63:312–336 (1988).
7 S. Ovsienko, Integral weakly positive forms, in *Schur Matrix Problems*, Kiev, 1978, pp. 3–17 (1978).
8 J. A. de la Peña, On the representation type of one-point extensions of tame concealed algebras, *Manuscripta Math.* 61:183–194 (1988).
9 J. A. de la Peña, Quadratic Forms and the Representation Type of an Algebra, Sonderforschungsbereich 343, 90-003, Univ. Bielefeld, 1990.
10 J. A. de la Peña, Algebras with hypercritical Tits form in *Topics in Algebra*, Banach Center Publ. 26, part I, 1990, pp. 353–369.
11 J. A. de la Peña, On the corank of the Tits form of a tame algebra, To appear in *J. Pure Appl. Alg.*
12 C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, *Lecture Notes in Math*, Springer-Verlag, 1984.
13 M. Zeldich, A criterion for weakly positive quadratic forms. in *Linear Algebra and Representation Theory*, Kiev, 1983.

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